POHEA Solutions

Don Edwards and Mike Syphers

October 2, 2007

Introductory Remarks

We turned in the material for publication of our book *Introduction to the Physics of High Energy Accelerators*, or POHEA for short, fifteen years ago in the Summer of 1992. The original plan was to write up the solutions to the problems back then, but one of us had already gone to work at DESY and the other would soon depart to BNL. However the intent stayed in the queue, and this year we decided to act on it. There have been occasional requests but we also wondered if we could still do the problems.

The end-of-chapter problems were based primarily on our experience at Cornell and Fermilab in the design, construction, commissioning and operation of the synchrotrons at those two institutions. That is why the emphasis in the book is on single particle dynamics as applied to synchrotrons. It also accounts for the lack of emphasis on the acceleration process itself, although we did realize that a book about accelerators should at least mention RF systems before going on to transverse stability.

It's an embarrassment to see errors (beyond the usual mention of typos) in the problem statements. Some of these we might try to attribute to "time pressure" or the like, but how can there be a possible excuse for the association of the matrix designated as M_0 in Problem 19 of Chapter 3 with the intent of the exercise. That is the most egregious stupidity that we have detected but there are others, as we comment in succeeding sections for the chapters.

There are two main things missing from the book when we look at it in today's context. One is the impact of superconducting RF systems and the anticipation that an electron-positron collider based on that technology will be the HEP follow-on to the LHC. The other is the dramatic increase in the application of synchrotron radiation to research in biology, chemistry, and physics and the accelerator developments associated with those directions.

1 Solutions: Chapter 1

Problem 1 The mass (in energy units) of a proton is 0.938 GeV = 1.5×10^{-10} joules, so for this 1 joule particle $\gamma \approx 6.7 \times 10^9$. Then using Eq. 1.3

$$\begin{aligned} \frac{1}{\gamma^2} &= \frac{c^2 - v^2}{c^2} = \frac{(c+v)(c-v)}{c^2} \approx 2\frac{c-v}{c}, \\ c - v &= 3.4 \times 10^{-12} \ \ \text{m/sec} \end{aligned}$$

Problem 2 Transform $x' = u't'$ according to Eqs. 1.15 and 1.18, set $x = ut$, and solve for u. The result is

$$u = \frac{u' + v}{1 + u'v/c^2}$$

Problem 3 The time interval dt' experienced by the passenger is related to the laboratory time interval dt by $dt' = dt/\gamma = ds/(v\gamma)$, where s is the position along the path of total length $L = 2$ miles. After expressing v and γ in terms of s, the total time T' is

$$T' = \int_0^L \frac{ds}{c\left[2(\gamma_F - 1)\frac{s}{L} + (\gamma_F - 1)^2\frac{s^2}{L^2}\right]^{1/2}}$$

where $\gamma_F = 50 \times 10^3/0.51 = 0.98 \times 10^5$. This integral involves hypergeometric functions, so it's easier to do it numerically. The result is $T' = 0.71$ nanoseconds. If one uses the approximation $v = c$ over the entire path, then the integral is elementary and yields $T' = 0.67$ nanoseconds.

Problem 4 Using Eq. 1.1 on page 4,

$$dW = \vec{F} \cdot d\vec{s} = (d\vec{p}/dt) \cdot d\vec{s} = \vec{v} \cdot d\vec{p} = mv^2 d\gamma + m\gamma dv^2/2 = mc^2 d\gamma.$$

Problem 5 Starting from $\vec{p} = \gamma m\vec{v}$ following Eq. 1.1, proceed as follows:

$$p^2c^2 = m^2c^4\gamma^2\frac{v^2}{c^2} = m^2c^4\gamma^2\left(1 - \frac{1}{\gamma^2}\right) = E^2 - m^2c^4$$

which upon rearrangement yields the result.

Problem 6 Using the result of the preceding problem

$$\frac{dE}{E} = \frac{c^2 p dp}{\gamma^2 m^2 c^4} = \frac{p^2}{\gamma^2 m^2 c^2}\frac{dp}{p} = \frac{v^2}{c^2}\frac{dp}{p}.$$

Problem 7 For part (a), use Eq. 1.11 with a current $\lambda' u'$ integrated around a circle of radius r' to give the result for $B'_\perp$ as stated. The statement of part (b) is unintentionally misleading; the word "dividing" should not have been used; both multiplicaton and division are involved. Also, it would have been helpful if we had included back in Problem 2 another relationship about the addition of velocities, namely for a speed u' then back in the lab we have

$$\gamma(u) = \gamma(u')\gamma(v)\left(1 + \frac{u'v}{c^2}\right).$$

The current in the laboratory system is

$$-\gamma(v)\lambda' v + \gamma(u')\gamma(v)(u' + v)\frac{\lambda'}{\gamma(u')}$$

where use has been made of the preceding expression for $\gamma(u)$. Integration as in part (a) gives the relation between $B_\perp$ and $B'_\perp$. The net charge per unit length in the lab frame is

$$-\gamma(v)\lambda' + \gamma(u')\gamma(v)(1 + u'v/c^2)\frac{\lambda'}{\gamma(u')}$$

and now apply Eq. 1.8 to a cylinder of unit length and radius $r = r'$ to obtain the radial electric field. Recall that $c^2 = \mu_0 \varepsilon_0$.

Problem 8 Equating the forces, $evB = mg$, from which

$$v = \frac{mg}{eB} \approx 1 \ \text{mm/sec}$$

and since this is far from a relativistic speed, the kinetic energy is

$$T = \frac{1}{2}mv^2 \approx 10^{-33} \ \text{J}$$

Problem 9 A static electric field is related to the electrostatic potential V according to $\vec{E} = -\vec{\nabla}V$. So averaging over the surface of a sphere of radius R we have from Eq. 1.8

$$\frac{1}{4\pi R^2}\int \vec{E}\cdot d\vec{S} = \frac{1}{4\pi}\frac{d}{dR}\int V d\Omega = 0$$

where $d\Omega$ represents the solid angle of the integral over the surface of the sphere. Extension of this expression down to the limit of zero radius given that V is supposed to be a continuous function yields $V(0)$ at $R = 0$.

Problem 10 This exercise outlines the so-called Relaxation Method for numerical solution of a set of boundary value problems. Very time-consuming before the advent of computers.

Problem 11 Use the static version of Eq. 1.12. In the infinite permeability approximation, $\vec{H}$ vanishes within the iron, so the only contribution to the line integral comes from the gap of dimension h.

Problem 12 Integration of $\partial^2\Phi_m/\partial x\partial y = B'$ takes care of part (a). At constant x, use Eq. 1.12 to integrate B_y from one pole to the other, then close the line integral by looping through the steel around two of the coils. The hyperbolic pole faces are of the form $xy = R^2/2$. Then

$$\oint B_y dy = B'x \int_{-R^2/2x}^{R^2/2x} dy = B'R^2 = \mu_0 2NI$$

and solving for B' gives the result.

Problem 13 Since the divergence of a gradient is ∇^2 and the divergence of B is zero, Φ_m satisfies Laplace's equation. The forms shown in parts (a) and (b) follow.

Problem 14 Referring to the figure, imagine a triangle with two of its apexes at the centers of the conductors and calculate the field at a third apex within the overlap region. Suppose the current density in the conductor at the left is coming toward you. Then if the distance from its center to the third apex is r_1, the magnitude of the magnetic field is $B_1 = \mu_0 J r_1/2$ after application of Eq. 1.11. Similarly, the magnitude of the field due to the conductor at the right is $B_2 = \mu_0 J r_2/2$. If we call the included angles at the left and right vertices of the triangle ϕ_1 and ϕ_2, then the vertical component of the field is $B_1 \cos\phi_1 + B_2 \cos\phi_2$. But putting in the values of B_1 and B_2, the term $r_1 \cos\phi_1 + r_2 \cos\phi_2$ is just the distance d between the conductor centers. So the vertical component of the magnetic field is a constant, $\mu_0 J d/2$, within the overlap region. The horizontal component is zero because $r_1 \sin\phi_1 = -r_2 \sin\phi_2$.

Problems 15 and 16 These two problems go a bit beyond the basic emphasis of this chapter. For the introduction of the overlap integral and many ways in which luminosity can be expressed, see the article by Furman and Zisman in Section 4.1 of the Chao-Tigner Handbook.

Problem 17 At a density of 0.07 gm/cm^3, and given the proton mass of 1.7×10^{-24} gm, there are about 4×10^{22} targets per cm^3. So in the 1 m length of the target, the fraction of the area that is presented to a passing proton is about $4 \times 10^{25}\sigma_{int}$. For a proton flux of 10^{11} per second, the luminosity would be 4×10^{36} cm^{-2} sec^{-1}.

2 Solutions: Chapter 2

Problem 1 Eq. 2.1 relates the radius of curvature to the magnetic field. The ratio of momentum/charge is called the magnetic rigidity and has the dimensions of tesla-meters; it's discussed in the next chapter. But might as well introduce it here. The product of field and radius is 10/3 times the momentum in GeV/c, so the field is 1 tesla. For a betatron, the flux through the orbit would be, according to Eq. 2.3, 2π tesla-m^2, or a 1 meter yoke radius at saturation. The 300 MeV betatron at the University of Illinois in the 1950s was a truly impressive device among electron accelerators of that era.

Problem 2 Use of the definition of surface resistivity in the lines following Eq. 2.17 gives the result stated. At $f = 400$ MHz in copper, taking $\rho = 1.8 \times 10^{-6}$ ohm-cm, gives $\rho_s = 5.3 \times 10^{-3}$ ohms.

Problem 3 Manipulation of Eq. 2.23 results in the form

$$R_s = A\frac{\sin^2(\alpha x)}{x+1}$$

where $x \equiv L/R$, $\alpha \equiv 1.2/(v/c)$, and the constant A out in front is everything left over that plays no role in the following. The maximum of R_s as a function of x for $\alpha = 1.2$ gives $L/R \approx 1.15$.

Problem 4 At 400 MHz, a quick numerical estimate shows that E_k is around 20 MV/m; putting on a factor of 1.7 gets the gradient up to 34 MV/m. If you elect for a cell for which the passage corresponds to one-half an oscillation, a so-called π-mode cell, then the transit time factor is $2/\pi$, and so the energy gradient for each cell could be as high as 22 MeV/m. At a cell length of 3/8 m ($v \approx c$), some 8 MeV could be achieved in a single cell. As noted in the text, the cell radius is about 0.3 m, and so all the numbers are in hand to calculate the power dissipation, etc.

Problem 5 In Eq. 2.40, insert $\phi_1 = \pi - \phi_s$. Then $d\Delta E/dn = 0$, and ϕ_1 is one the boundaries of the motion in ϕ. Then looking at Eq. 2.45, the other boundary is given by the relationship stated in the problem.

Problem 6 This exercise may be found under Lecture Demonstrations on this site.

Problem 7 At the injection energy of 2 TeV, the proton is highly relativistic, so its speed may be taken as c. The rate of energy gain is 12×10^3 MeV/sec, which at an orbit period of 0.29×10^{-3} seconds implies a rate of energy increase of 3.5 MeV per turn. So $\sin(\phi_s) = 3.5/15$ and $\phi_s = 167^o$. Then using Eq. 2.49 $\nu_s \approx 0.11$ at injection. This result of "10 turns per synchrotron oscillation" at injection is quite typical of high energy synchrotron designs.

Problem 8 Suppose the situation is above transition so $\phi_s = \pi$. Then Eq. 2.45 takes the form

$$\Delta E^2 + A\cos\phi = A$$

where the "constant" on the right hand side of the equation has been evaluated at $\phi = 0$ where $\Delta E = 0$. Then

$$\int \Delta E dt = \frac{1}{\omega_{rf}} \int_0^{2\pi} \Delta E d\phi = \frac{1}{\omega_{rf}} 4\sqrt{2A}$$

yields the $\Delta E > 0$ half of the area. Multiplication by 2 gives the result. A similar calculation below transition has the same outcome. The bucket area associated with the (Tevatron) parameters is 7.1 eV-sec.

Problem 9 The area is supposed to be occupied by particles executing small oscillations above transition, with a maximum amplitude of $\phi_m = 0.5$. Moving the center of oscillation to π, Eq. 2.45 becomes

$$\Delta E^2 = A(\cos\phi - \cos\phi_m) \approx \frac{A}{2}(\phi_m^2 - \phi^2)$$

and the result follows from integration over $\Delta E dt$ in the above approximation. The emittance is 0.35 eV-sec.

Problem 10 According to Eqs. 2.71-2.72, dp/p will damp like $1/E^{3/4}$ well above transition and like $1/E^{1/4}$ well below (provided $v \approx c$).

Problem 11 A bit more algebra here for special function enthusiasts. Take a look at for instance RHIC Technical Note No. 7 by Harold Hahn dated August 28, 1984 entitled "The Stationary RF Bucket", which can be downloaded from Brookhaven.

Problem 12 This exercise may be found under Lecture Demonstrations on this site.

Problem 13 This is Eq. 2.45 again, with $\Delta E/E$ evaluated at $\phi = \pi$. Using the notation for the solution to Problem 8 above, $\Delta E/E = \sqrt{2A}/E = 5 \times 10^{-3}$.

Problem 14 One approach would be to use two traveling wave accelerating structures, in recognition that particles going in the opposite direction to such a wave are perturbed very little. Or if the designers have a preference for standing wave cavities, like the pillboxes described in the text, two sequences of such cavities with relative phases chosen to reinforce acceleration in the two directions will perform the same function.

3 Solutions: Chapter 3

Problem 1 This exercise is supposed to reinforce the discussion of Section 3.1.1, and weak focussing played an important role in the history. The results can be obtained using Eqs. 3.46-3.47, although somehow the name of the vertical coordinate differs between that used in Eq. 3.47, y, and the z of the problem statement.

Problem 2 Unfortunately, the point of this problem was not stated clearly. The intent was to extend the discussion of page 62 associated with Eq. 3.11 to larger ratios of $L/f \ll 1$ to show that a quadrupole doublet was a useful focusing device in beam optics. To pursue this, see how the familiar single lens formula of geometrical optics $1/\ell_1 + 1/\ell_2 = 1/f$ carries over into this case. For the doublet be useful in replacement of a single (convex) lens, one would be looking at the $\ell \gg L$ range which could be of interest in beam lines, but not of relevance for accelerator structures.

Problem 3 A past approach to thick lens description replaced the thick lens with a drift1-thin lens-drift2-thin lens-drift1 sequence, where the locations of the two thin lenses were referred to as the principal planes. With today's computer capability, this description and hence this exercise is of limited interest. See, for example, http://en.wikipedia.org/wiki/Focal_plane.

Problem 4 Here, the point is to reproduce Eq. 3.21. Just remember that if the matrix is to operate on a column vector, the order of factors builds from right to left.

Problem 5 After multiplication of Eqs. 3.50 and 3.51, the stability criterion, Eq. 3.20, requires $|\cosh(\sqrt{K}L)\cos(\sqrt{K}L)| \leq 1$.

Problem 6 Computer exercise.

Problem 7 As shown in the progression from Eq. 3.15 through Eq. 3.19, neither eigenvalue is real but the sum is. Similarly, neither eigenvector is real, but their superposition in description of physical motion must be.

Problem 8 In one degree-of-freedom, unimodularity of the transfer matrix is the equivalent of conservation of energy in this description of the motion. For an oscillator, a quadratic form like $ax^2 + bx'^2$ should be conserved. If M is the matrix to convey the motion from point 1 to point 2

$$(ax^2 + bx'^2)_2 = \begin{pmatrix} ax & bx' \end{pmatrix}_1 M^T M \begin{pmatrix} ax \\ bx' \end{pmatrix}_1 = (ax^2 + bx'^2)_1$$

therefore $M^T M = 1$ and since the determinant of the transpose is the same as the determinant of the matrix, the square of the determinant is one. Matrices with determinants equal to -1 are concerned with reflections, therefore in the present case we would expect a unit determinant. In the x, x' description the determinant will change with energy; the use of "invariant emittance" is supposed to help in this regard.

Problem 9 Best that we set aside for the moment why one would choose to write the matrix in this form; it probably made some sort of sense back when we were writing the book. This is rather formal material. One can write $M = e^{J\mu}$, and that exhibits the similarity of the J matrix in this context to $\sqrt{-1}$. With $K = J\mu$ the circumstance that the trace of K is zero follows immediately.

Problem 10 Just multiply the matrices. This is a useful result.

Problem 11 The rather formidable looking Eq. 3.72 becomes tractable if differentiated once more, and becomes linear. This is an important result; the amplitude function is no more complicated than an offset harmonic oscillator in progress through an accelerator component.

Problem 12 No explanation is necessary here; the results are shown.

Problem 13 This expression is basic to the analysis of orbit perturbation in beam lines or accelerator structures, presuming that an initial design has been developed in terms of Courant-Snyder parameters. In Eq. 3.71, evaluate the constants A, δ with the initial conditions $x = 0, dx/ds = \theta$.

Problem 14 Since the phase advance around a ring is the same independent of position,

$$M_2 = I\cos\mu + J_2\sin\mu = M(1,2)[I\cos\mu + J_1\sin\mu]M(1,2)^{-1}$$

and the result follows after cancellation of the $\cos\mu$ terms. The detailed relationships between Courant-Snyder parameters follows from straightforward matrix multiplication.

Problem 15 This result follows from manipulation of the first row of Eq. 3.85.

Problem 16 In a drift space, the amplitude function is a parabola; that was one of the results of Problem 11. If we call ϕ the phase advance in a distance L then its sine is

$$\begin{aligned}\sin\phi = \sin\left[\int_0^L \frac{ds}{\beta_0 - 2\alpha_0 s + \gamma_0 s^2}\right] &= \sin[\arctan(\gamma_0 L - \alpha_0) + \arctan\alpha_0] \\ &= \frac{\gamma_0 L}{[1 + (\gamma_0 L - \alpha_0)^2]^{1/2}[1 + \alpha_0^2]^{1/2}}\end{aligned}$$

and this is the quantity that is to be compared with the result obtained directly from the trace of the matrix. The expressions for the necessary Courant-Snyder parameters in terms of lens spacing and focal length may be obtained by comparing the form of the J matrix with Eq. 3.21, with the results

$$\alpha_0 = \left[\frac{1+\frac{L}{2f}}{1-\frac{L}{2f}}\right]^{1/2}, \quad \gamma_0 = \frac{1}{f\left[1-\left(\frac{L}{2f}\right)^2\right]^{1/2}}$$

These yield $\sin\phi = L/2f$, consistent with the value found from the trace.

Problem 17 The amplitude function at the collision point is conventionally denoted by β^*, and since β is a minimum there, $\alpha = 0$, $\gamma = 1/\beta^*$, and the phase advance moving through the drift region in either direction is

$$\psi(s) = \int_0^s \frac{dz}{\beta^* + \frac{z^2}{\beta^*}} = \arctan(s/\beta^*).$$

For $s \gg \beta^*, \psi \approx \pi/2$, and so the total phase advance approaches 180°.

Problem 18 To show that $J^2 = -I$, carry out the multiplication and use Eq. 3.68. The n repetition part can be proved by induction, or by an appeal to Eq. 3.76.

Problem 19 There is an (embarrassing) error in the statement of the problem. As shown, with 0's on the main diagonal, the motion is about as safely stable as it can get, since the tune is a quarter of an integer. The way it *should* have gone is as follows. If the tune of stable motion differs from an integer by a small amount ε the matrix at a point where the amplitude function is a maximum and its derivative will be 0 takes the form

$$\begin{pmatrix} 1-\frac{\varepsilon^2}{2} & \frac{\beta_0}{\varepsilon}\varepsilon \\ \frac{\varepsilon}{\beta_0} & 1-\frac{\varepsilon}{2} \end{pmatrix} \overset{\varepsilon \to 0}{=} \begin{pmatrix} 1 & \beta_0 \\ 0 & 1 \end{pmatrix}$$

where β_0 is the maximum of beta near a quarter integer tune. Then the introduction of a thin defocusing lens of focal length $\beta_0/4$ gives

$$\begin{pmatrix} 1 & 0 \\ \frac{1}{4\beta_0} & 0 \end{pmatrix}\begin{pmatrix} 1 & \beta_0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \beta_0 \\ \frac{1}{4\beta_0} & \frac{5}{4} \end{pmatrix}.$$

Then, using Eq. 3.79, $\cos\mu = 1 + 1/8$ which yields $\sin(\mu/2) = i/4$; that is, the phase advance has become imaginary. Of course, because the variables $x, dx/ds$ are real numbers, the amplitude function also turns imaginary at $1.9i\beta_0$.

Problem 20 From Problem 12, and recalling that in this case $\gamma_{max} = 1/\beta_{min}$, $\beta_{max} = 101$ m and $\gamma_{max} = 29$ m. Then Eqs. 3.106 and 3.107 give 14 mm and 0.26 milliradians for the estimates.

Problem 21 Using Eq. 3.95, admittance $= 25\pi$ mm-mrad.

Problem 22 Suppose there is a large number P of synchrotrons in the ensemble. Starting from Eq. 3.146, the average of the square of the displacement at some point will be

$$\begin{aligned} \langle x^2 \rangle = \frac{1}{P}\sum_{\ell} x_\ell^2 &= \left(\frac{\beta_0^{1/2}}{2\sin\pi\nu}\right)^2 \sum_{i,j}\left[\frac{1}{P}\sum_{\ell}\theta_{i,\ell}\theta_{j,\ell}\right](\beta_i\beta_j)^{1/2}\cos(\pi\nu-\psi_i)cos(\pi\nu-\psi_j) \\ &= \left(\frac{\beta_0^{1/2}}{2\sin\pi\nu}\right)^2 \sum_{i}\theta_{\rm rms}^2\beta_i\cos^2(\pi\nu-\psi_i) \end{aligned}$$

where ℓ is the index designating a member of the ensemble, and β_0 is the amplitude function at the location where the total displacement is calculated. The quantity in square brackets for uncorrelated perturbations reduces to δ_{ij}, and that has been used to yield the second line. Also in the second line, note that the square of the cosine appears rather than the sine as in the statement of the problem. Apparently there was some sort of transcription error. The final result follows from

$$\sum_i \beta_i\cos^2(\pi\nu-\psi_i) = \sum_i \beta_i\left(\frac{1+\cos[(\pi\nu-\psi_i)/2]}{2}\right) \approx \frac{N\overline{\beta}}{2}$$

The numerical part of this problem was based on the Superconducting Supercollider (SSC) parameters. Assuming 5 Tesla magnets, the circumference would be about 85 km or some 500 FODO cells. The quadrupole focal length is $90/\sqrt{2} = 63$ m giving $\theta_{\rm rms} = 0.016$ mrad. Using the results of Problem 12, $\overline{\beta} = 180$ m and $\beta_0 \approx 300$ m. Then the estimate of the rms orbit distortion is 3 cm.

Problem 23 Computer exercise.

Problem 24 The requirement that displacement and slope be zero following θ_3 leads to the conditions

$$\begin{aligned} x &= \theta_1(\beta_1\beta_3)^{1/2}\sin\psi_{13} + \theta_2(\beta_2\beta_3)^{1/2}\sin\psi_{23} = 0 \\ x' &= \theta_1\left(\frac{\beta_1}{\beta_3}\right)^{1/2}[\cos\psi_{13}-\alpha_3\sin\psi_{13}] + \theta_2\left(\frac{\beta_2}{\beta_3}\right)^{1/2}[\cos\psi_{23}-\alpha_3\sin\psi_{23}] + \theta_3 = 0 \end{aligned}$$

Solving for θ_2 and θ_3 in terms of θ_1 gives the results.

Problem 25 Now is the time to use that matrix erroneously stated in Problem 13! The focal length of the quadrupole in that very stable situation is about $\beta_0/4$ where β_0 is the maximum value of the amplitude function. Reversal of the quadrupole makes the perturbation twice as strong, so the matrix becomes

$$\begin{pmatrix} 1 & 0 \\ \frac{8}{\beta_0} & 1 \end{pmatrix}\begin{pmatrix} 0 & \beta_0 \\ -\frac{1}{\beta_0} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \beta_0 \\ -\frac{1}{\beta_0} & 8 \end{pmatrix}.$$

With $\cos\mu = 4$, the motion is not stable.

Problem 26 This exercise is in the same spirit as Problem 22 but there is less algebra because no trigonometric functions are involved. Start from Eq. 3.151, and let q_i stand for the unperturbed $1/F$ of the ith quadrupole. Again, assume the ensemble contains a number P of nominally identical synchrotrons, and the index ℓ identifies a member of that ensemble. Then

$$\begin{aligned}\langle\Delta\nu^2\rangle &= \left(\frac{1}{4\pi}\right)^2 \sum_{i,j}\left[\frac{1}{P}\sum_{\ell}\left(\frac{\Delta B'}{B'}\right)_{i,\ell}\left(\frac{\Delta B'}{B'}\right)_{j,\ell}\right] q_i q_j \beta_i \beta_j \\ &= \left(\frac{1}{4\pi}\right)^2 \left(\frac{\Delta B'}{B'}\right)^2_{\text{rms}} N_{\text{cell}}\frac{\beta^2_{\text{max}}+\beta^2_{\text{min}}}{F^2}.\end{aligned}$$

For the numerical parameters of this problem, the rms tune spread of the ensemble is about 0.01, which looks small but is an order of magnitude too large for comfort due to the non-linear effects of Chapter 4.

Problem 27 Using the results of Problem 14 above, the unperturbed amplitude function downstream of the quadrupole can be written as $\beta_0(s) = a^2\beta_1 - 2ab\alpha_1 + b^2(1+\alpha_1^2)/\beta_1$, where a and b, which are functions of s, are the 1,1 and 1,2 elements of the matrix from the quadrupole to the point downstream. The introduction of the quadrupole error changes the initial value of α according to Eq. 3.153, so that the new amplitude function must follow $\beta(s) = a^2\beta_1 - 2ab(\alpha_1 + q\beta_1) + b^2(1 + (\alpha_1 + q\beta_1)^2)/\beta_1$. Subtracting the two expressions above, and using Eq. 3.85 to write $a(s)$ and $b(s)$ in terms of the unperturbed amplitude functions and phase advance, generates the stated result.

Problem 28 The result is obtained by an analogous procedure to that used in Problems 22 and 26. In the form shown, the angle brackets have no effect other than to confuse the reader. Expressed in the form where the brackets make sense, the solution becomes

$$\left(\frac{\Delta\beta}{\beta}\right)_{\text{rms}} = \frac{1}{2\sqrt{2}|\sin 2\pi\nu]} N^{1/2}\langle q^2\beta^2\rangle^{1/2}$$

Problem 29 This problem can be easily done with a sketch, but here's an approach using two-by-two matrices. Go through a half-cell, starting at the midpoint of the focusing quadrupole where the dispersion will be a maximum. Midway through the drift, add the angle due to the dipole, then finish at the midpoint of the defocusing quadrupole where the dispersion is a minimum.

$$\begin{pmatrix} D_{\text{min}} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1/(2F) & 1 \end{pmatrix}\begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix}\left[\begin{pmatrix} 0 \\ \theta \end{pmatrix} + \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/(2F) & 1 \end{pmatrix}\begin{pmatrix} D_{\text{max}} \\ 0 \end{pmatrix}\right]$$

This yields two equations for the two unknowns, and with use of $\sin(\mu/2) = L/(2F)$ gives the result stated.

Problem 30 Let D, D' stand for the dispersion and its slope for the FODO cell alone, and $\Delta D, \Delta D'$ for the addition needed as a result of the bend-free cell. Closure through the n-cells requires

$$M^n \begin{pmatrix} \Delta D \\ \Delta D' \end{pmatrix} + M \begin{pmatrix} D \\ D' \end{pmatrix} = \begin{pmatrix} D + \Delta D \\ D' + \Delta D' \end{pmatrix}$$

To obtain the solution as stated, use

$$(I - M^n)^{-1} = \frac{I - M^{-n}}{2 - \mathrm{Trace} M^n}$$

and Eq. 3.19.

Problem 31 A particle that is initially on and directed along a displaced equilibrium orbit will stay on it. That is

$$\begin{pmatrix} D_2 \frac{\delta p}{p} \\ D_2' \frac{\delta p}{p} \\ \frac{\delta p}{p} \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_1 \frac{\delta p}{p} \\ D_1' \frac{\delta p}{p} \\ \frac{\delta p}{p} \end{pmatrix}$$

and the result follows from rearrangement.

Problem 32 With the abbreviations $c \equiv \cos\mu$ and $s \equiv \sin\mu$, the matching requirement may be written

$$\begin{pmatrix} c & \beta s & x(1-c)D \\ -s/\beta & c & xsD/\beta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c & \beta s & (1-x)(1-c)D \\ -s/\beta & c & (1-x)sD/\beta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

In the above equation, the cell matrices have been written from quadrupole center to quadrupole center where $D' = 0$. The result follows by solving for x. In the particular case $\mu = \pi/3$, matching could be achieved by leaving the standard dipoles out of the next to the last cell.

Problem 33 Estimate $\langle D/\rho \rangle$ using the result of Problem 29. Using the approximation $\sin(\mu/2) \approx \mu/2$,

$$\left\langle \frac{D}{\rho} \right\rangle \approx \frac{\theta^2}{\mu^2} \approx \frac{1}{\nu^2}$$

and the result follows from Eq. 3.136. In the above, use has been made of $\theta = 2\pi/2N$ and $N\mu = 2\pi\nu$, where N is the number of cells.

Problem 34 The lowest order contribution to the path length change is $\int (x ds/\rho)$. Throughout most of the circumference, the oscillatory motion will make a very small contribution to the integral; only in the region of the "kink" caused by θ will the motion stay on one side of the orbit center for more than a half-wavelength. Starting from Eq. 3.146, the series of approximations goes as follows:

$$\Delta C \approx \theta \frac{\beta_0^{1/2}}{2\sin(\pi\nu)} \int_0^{2\pi\nu} \frac{\beta^{3/2}}{\rho} \cos(\psi - \pi\nu) d\psi \approx \theta \frac{\beta^2}{\rho} \approx \theta \frac{L^2}{\rho \sin(\mu/2)^2} \approx \theta D$$

The phrase "at the location" in the statement of the problem is something of an exaggeration; "in the neighborhood" might be more appropriate.

Problem 35 Start from Eq. 3.147 and use the result of Problem 12.

$$\xi = -\frac{1}{4\pi} 2L \left[\frac{1+\sin(\mu/2)}{F\sin\mu} - \frac{1-\sin(\mu/2)}{F\sin\mu} \right] N_{\text{cells}} = -\frac{N\tan(\mu/2)}{\pi} \approx -\nu$$

Problem 36 For a "horizontal" oscillation, Eq. 3.155 gives

$$\delta\nu = \frac{1}{4\pi} \oint \frac{2B_0 b_2}{B_0 \rho} \beta D ds = \langle D\beta b_2 \rangle \frac{\delta p}{p}$$

and a minus sign enters in the "vertical" plane because $\nabla \times B = 0$.

Problem 37 Eqs. 3.163-4 follow from application of the result of the preceding problem. For a 20 TeV ring, from Problem 29 $D \approx 4$ m and from Problem 22 $\beta \approx 180$ m, so the chromaticity will have a magnitude of about 2×10^3.

Problem 38 The only challenge in this exercise is figuring out where d and L were supposed to be on the sketch. The offset sideways is (intended to be) d and the dipole spacing along the beam is L, so the dispersion function at B is d, which in the case of part (b) is 22.5 inches. Vertical emittance dilution will occur. For $\delta p/p \approx 10^{-3}$ the vertical spread from dispersion would be about 0.6 mm, which is comparable to the rms beam size due to transverse emittance alone.

4 Chapter 4

Problem 1 If magnetic fields other than those in the basic design are added to Eq. 3.48, the result is

$$x'' + K(x) = -\frac{\Delta B}{(B\rho)}$$

Then transform to the dependent variable $\zeta \equiv x/\beta^{1/2}$ using $\phi = \psi/\nu$ as the independent variable for ζ while maintaining x as the independent variable in β and its derivatives. Eq. 3.72 will be needed to simplify the result.

Problem 2 In the solution to Problem 11 of Chapter 1, replace h by g and then differentiate with respect to the transverse coordinate x to calculate B'. Use of $dg/dx \approx h/x$ gives the result. This approach continues to assume that $\vec{H} = 0$ within the iron.

Problem 3 Computer exercise.

Problem 4 Try "backing into" a Hamiltonian by using Eq. 4.49 to integrate Eq. 4.46 with respect to x:

$$\int -\frac{d\mathcal{H}}{d\tilde{x}} = -\frac{1}{4}A\tilde{p_x}^2 x + \frac{1}{4}A\frac{1}{3}x^3 + \pi\delta\tilde{x}^2 + f(\tilde{p_x})$$

where $f(\tilde{p_x})$ is some as yet unknown function of the other variable. Differentiation of this intermediate result with respect to $f(\tilde{p_x})$ as in Eq. 4.48 results in $f(\tilde{p_x}) = \pi\delta\tilde{p_x}^2$. Now there is a function that qualifies as a Hamiltonian; call it $\mathcal{H}_1$:

$$\mathcal{H}_1 = \pi\delta\tilde{p_x}^2 + \pi\delta\tilde{x}^2 - \frac{1}{4}A\tilde{p_x}^2 x + \frac{1}{4}A\frac{1}{3}x^3$$

Though it may not look like the invariant of Eq. 4.47, $\mathcal{H}_1$ yields the proper equations of motion. Eq. 4.47 is a modification contrived to exhibit the fixed points explicitly, and the equations of motion are just the same as would be obtained with Hamilton's equations on $\mathcal{H}_1$. So Eq. 4.47 is the preferred form of the Hamiltonian in this case, and could be arrived at directly by first identifying the fixed points from the equations of motion and making a coordinate change.

Problem 5 Identify y with $\tilde{p_y}$ and recall that $\int[1/(x^2 - a^2)] = [1/(2a)][\ln(x-a) - \ln(x+a)]$.

Problem 6 Three turns later, $y = 15$ mm. This is a rather realistic picture of the amplitude growth needed to direct particles into an extraction channel.

Problem 7 Computer exercise.

Problem 8 Computer exercise.

Problem 9 Computer exercise.

Problem 10 The expansion of the magnetic field as a function of x looks like $B(x) = B(0) + B'x + B''x^2/2 + B'''x^3/6 + ...$ so the B''' term represents the octopole field. If B''' has no significant dependence on s then Eq. 4.33 gives $da/dn = 0$. But insertion of the octopole field into Eq. 4.35 produces a $\cos^4 \chi$ term which expands to $3/8 + \cos 2\chi/2 + \cos 4\chi/8$ and the first term in this expansion results in a non-zero average for the tune shift as a function of amplitude.

Problem 11 To be consistent with the notation in the text, the equations of motion in the statement of the problem should contain $\tilde{x}$ and $\tilde{p_x}$. Use of the simpler notation was probably motivated by typing fatigue. We continue with the simpler form here. Starting from Eqs. 4.33 and 4.35, the equivalents of Eqs. 4.41 and 4.42 become

$$\begin{aligned} \frac{da}{dn} &= \frac{a}{2} Q \sin 2\psi \\ \frac{d\psi}{dn} &= \frac{1}{2} Q \cos 2\psi + 2\pi\delta \end{aligned}$$

Next, use Eqs. 4.43 and 4.44 to convert to x and p_x to give the equations of motion as shown in the problem statement. Some trigonometric maneuvers are needed, involving $\tan\psi = -p_x/x$ and the sine and cosine functions associated with that triangle. Differentiation of the equation for dx/dn and use of the equation for dp_x/dn gives

$$\frac{d^2x}{dn^2} + \left[(2\pi\delta)^2 - Q^2/4\right] x = 0$$

and as long as the bracketed coefficient of x is positive the motion is stable. The half-width of the stopband follows from $|Q/2| = 2\pi\delta$.

Problem 12 With the use of the results of the preceding two problems, Eqs. 4.43 and 4.44 give

$$\begin{aligned} \frac{dx}{dn} &= p_x \left[-\frac{1}{2} Q + 2\pi\delta + \frac{3}{8} D(x^2 + p_x^2) \right] \\ \frac{dp_x}{dn} &= x \left[-\frac{1}{2} Q - 2\pi\delta - \frac{3}{8} D(x^2 + p_x^2) \right] \end{aligned}$$

As an alternative to the approach used in Problem 4, this time look at the fixed points first. Setting $dx/dn = dp_x/dn = 0$ yields four fixed points:

$$\begin{aligned} x &= 0, \;\; p_x = \pm 2 \left[\frac{Q - 4\pi\delta}{3D} \right]^{1/2} \equiv \pm u \\ p_x &= 0, \;\; x = \pm 2 \left[\frac{-Q - 4\pi\delta}{3D} \right]^{1/2} \equiv \pm v \end{aligned}$$

Suppose $B''' > 0$ so that the tune increases with amplitude. Then for stable motion of small amplitude oscillations we need $\delta < 0$ and $|Q| < 4\pi\delta$, so for extraction the phase stable region will shrink as $|Q|$ increases. Noting that $u > v$, there is reason to suspect that $\pm u$ are the centers of the circles in the statement of the problem, with radius $r = (u^2 + v^2)^{1/2}$. Indeed, a Hamiltonian given by

$$\mathcal{H} = \left[(p_x - u)^2 + x^2 - r^2\right]\left[(p_x + u)^2 + x^2 - r^2\right]$$

produces the correct equations of motion and $\mathcal{H} = 0$ is the product of two circles.

5 Chapter 5

Problem 1 Starting from $B_u = B'v$ and $B_v = B'u$, application of the two transformations

$$\begin{pmatrix} B_x \\ B_y \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} B_u \\ B_v \end{pmatrix}, \quad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

gives the result.

Problem 2 An ideal solenoid has a uniform field, B_s, in the beam direction s. Start from the equation of motion, $d\vec{p} = e\vec{v} \times \vec{B}$ and convert to s as the independent variable, with the result

$$\frac{dx'}{ds} = ky' \ , \quad \frac{dy'}{ds} = -kx' \ , \quad k \equiv \frac{B_s}{(B\rho)}$$

One quick way to solve this pair of equations is to combine them by defining $u' \equiv x' + iy'$ so one has $du'' = -iku'$ with the solution $u' = u_0' exp(-iks)$. Another integration gives the solution for u in terms of u_0 and u_0'. The matrix for the body of the solenoid is

$$M_{\text{body}} = \begin{pmatrix} 1 & s/k & 0 & -(c-1)/k \\ 0 & c & 0 & s \\ 0 & (c-1)/k & 1 & s/k \\ 0 & -s & 0 & c \end{pmatrix}$$

where $c \equiv \cos(k\ell), s \equiv \sin(k\ell)$, ℓ being the length of the solenoid. At exit from a (cylindrically symmetric) solenoid, a particle encounters a radially directed magnetic field. An attractively simple approximation is that the exit region is sufficiently short so that the displacement of a particle does not change. Looking downstream, a particle at radius r will receive a clockwise-directed momentum impulse of magnitude $\Delta p_\phi = ev_s \int B_r dt = e \int B_r ds$. But the longitudinal flux from a circle of radius r, $B_s \pi r^2$, is related to B_r by $2\pi r \int B_r ds$ due to $\int \vec{B} \cdot d\vec{A}$. The matrix becomes

$$M_{\text{exit}}(k) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -k/2 & 0 \\ 0 & 0 & 1 & 0 \\ k/2 & 0 & 0 & 1 \end{pmatrix}$$

and since at the entrance, B_r has the opposite sign, $M_{\text{enter}}(k) = M_{\text{exit}}(-k)$.

Problem 3 There is nothing fundamentally wrong with the linear optics of such a synchrotron; it's just inconvenient. The dispersion would be coupled into both the horizontal and vertical planes, and it is a bit difficult to think of an advantage in that. But that may be just a result of our lack of imagination.

Problem 4 Finding the eigenvectors means exhibiting the solutions for a from Eqs. 5.7 or 5.8. The result is

$$a = \frac{1}{2k}\left(k_1 - k_2 \pm \left[(k_2 - k_1)^2 + 4k^2\right]^{1/2}\right)$$

which follows rather directly from Eq. 5.7. The solution for a from Eq. 5.8 is the same, but some algebra is needed to demonstrate the equality.

Problem 5 This is a repetition of the technique for solving coupled linear differential equations that was used in Problem 2. Combination of Eqs. 5.49 and 5.50 gives as a next-to-last step

$$\frac{du}{dn} = -i\frac{\kappa}{2}\frac{u^*u}{(x^2+p_x^2)}v - i\sqrt{2}\pi\delta u$$

where u^* is the complex conjugate of u.

Problem 6 Computer exercise.

Problem 7 This is mainly a matter of substituting elements of Eq. 5.67 into Eq. 106, remembering to calculate a symplectic conjugate using Eq. 5.98.

Problem 8 In the statement of the problem, we should have also made clear that the derivatives of the amplitude functions were the the same in the two degrees of freedom. The intent was that the matrix between the two skew quadrupoles is the identity matrix. Then the product of the two skew quadrupole factors is also the identity matrix, and so the matrix describing motion outside of the skew quadrupole region is not modified by their insertion. The eigenvalues of the one-turn matrix are independent of the location at which they are calculated. To see that this is the case, look at the relationship between the one-turn matrices at two points: $M_2 = M(1,2)M_1M(1,2)^{-1}$. The eigenvalue equations at the two points are the same, as shown by

$$\begin{aligned}
\det\left[M_2 - \lambda I\right] &= \det\left[M(1,2)M_1M(1,2)^{-1} - \lambda M(1,2)M(1,2)^{-1}\right] \\
&= \det\left[M(1,2)\right]\det\left[M_1 - \lambda I\right]M(1,2)^{-1} = \det\left[M_1 - \lambda I\right]
\end{aligned}$$

Problem 9 Think of the matrix for the n degree-of-freedom system as an $n \times n$ array of elements m_{ij} each of which is itself a 2×2 matrix. The diagonal terms of the Eq. 5.88 product lead to n results analogous to Eqs. 5.94 and 5.95, except now each is the sum of n determinants. So these terms give n relationships. The off-diagonal terms occur in transposed pairs as in Eqs. 5.91 and 5.92, giving $2n^2 - 2n$ relationships. The sum of these two sources gives the $n(2n-1)$ answer.

Problem 10 Follow the treatment of Section 5.2 for the sextupole case, but this time the fields are $B_y = B'y$ and $B_x = -B'x$. Then the counterparts of Eqs. 5.115 and 5.116 are

$$\Delta x' = -\frac{B'}{(B\rho)}yds, \quad \Delta y' = -\frac{B'}{(B\rho)}xds$$

and the development follows just about line by line, except that now the phase differences are used in the trigonometric expansions because of the neighborhood of the difference resonance. Eq. 5.122 and the equivalent for rate of change of b become

$$\begin{aligned}
\frac{da}{dn} &= \frac{b}{2}\left[A\sin(\psi_x - \psi_y) + B\cos(\psi_x - \psi_y)\right] \\
\frac{db}{dn} &= -\frac{a}{2}\left[A\sin(\psi_x - \psi_y) + B\cos(\psi_x - \psi_y)\right]
\end{aligned}$$

where the "driving terms" A, B resemble those in Eqs. 5.123 and 5.124, though with phase differences rather than sums in the arguments of the sines and cosines. Division of one of these last equations by the other and integrating yields the result that $a^2 + b^2$ is constant. In contrast to the sextupole sum resonance, motion in this difference is bounded. Because of the phase difference between the two driving terms, two circuits of compensating skew quadrupoles would be needed for correction. The exception to this last statement is the case $m = 0$; for the zero harmonic usually one circuit is adequate.

Problem 11 This relationship can be obtained by using the result of Problem 22 of Chapter 3; the argument is essentially the same. A particle displaced in x at $D_x \delta p/p$ passing through a skew quadrupole of strength q will experience deflection through an angle $\theta = qD_x\delta p/p$, thus leading to vertical dispersion. For the numerical example, let's just use maximum value of the horizontal dispersion and maxima of both amplitude functions, thereby ignoring half of the quadrupoles and overestimating the effect of the other half:

$$\frac{D_y}{D_{\max}} \approx \frac{\beta_{\max}}{F} \frac{1}{|\sin(\pi\nu)|} \theta_{\text{rms}} (N/2)^{1/2} \approx 0.2$$

This estimate, though rough, already says that correction would be needed by skew quadrupole circuits.

Problem 12 Computer exercise.

6 Chapter 6

Problem 1 The force starts at zero, reaches a maximum at $r/\sigma = 1.6$ and then falls with increasing r. The derivative begins at its maximum with zero slope and negative curvature, falling off rapidly to about 20% of its initial value at $r = \sigma$.

Problem 2 The contribution of the space charge force to $d^2u/d\psi^2$ follows from

$$F_x = \frac{dp_x}{dt} = \gamma m \frac{d^2x}{dt^2} = \frac{\gamma m v^2 \sigma}{\beta^2}\frac{d^2u}{d\psi^2}$$

where use has been made of $d\psi = ds/\beta$. The remainder of Part (a) consists of using Eq. 6.18, the expression for ϵ_N associated with that equation to express the coefficient of the space charge force in terms of $\Delta\nu$, and the smooth approximation $\beta = R/\nu$. For Part (b), the left hand side of equation after substitution of the solution gives

$$u'' + u = \left[1 - (1+\delta/\nu)^2\right] u_0 \cos[(1+\delta/\nu)\psi] \approx -2\frac{\delta}{\nu} u_0 \cos[(1+\delta/\nu)\psi]$$

Moving the $u_0 \cos[(1+\delta/\nu)\psi]$ factor to the right hand side, the lowest order solution for δ is

$$\begin{aligned}
\delta(u_0) &= -\frac{\Delta\nu}{(u_0/2)^2}\left\langle \frac{1 - e^{-(\frac{u_0^2}{2})\cos^2\psi}}{\cos^2\psi}\right\rangle_\psi \\
&= -\frac{\Delta\nu}{(u_0/2)^2}\left\{1 - e^{-\frac{u_0^2}{2}}\left[I_0\left\{(u_0/2)^2\right)\right\} + I_1\left\{(u_0/2)^2\right)\right\}\right]\right\}
\end{aligned}$$

and, again in the spirit of a lowest order calculation, the contribution of I_1 is ignored for small u_0 in the result stated in the text.

Problem 3 Computer exercise.

Problem 4 The factor of γ^2 in the denominator of Eq. 6.18 comes from the opposite directions of the electric and magnetic forces. Without this factor, the tune shift would increase by 50% to 0.6.

Problem 5 Computer exercise.

Problem 6 Using the sequence Eq. 1.23, Table 3.1, Eq. 3.119, Eq. 6.21 gives

$$\mathcal{L} = f\frac{N^2}{4\pi\sigma^2} = f\frac{N^2}{4\varepsilon\beta^*} = f\frac{\gamma N^2}{4\varepsilon_N\beta^*} = \frac{cn_B}{2\pi R}\frac{\gamma\Delta\nu N}{r_0\beta^*} = 1.3\times 10^{31}\ \mathrm{cm^{-2}s^{-1}}$$

where n_B is the number of bunches and at 1 TeV the proton speed is taken to be the speed of light in calculating the bunch collision frequency.

Problem 7 The transition is shown in the solution of the preceding problem when the N^2 dependence changes to N with the insertion of the tune shift parameter.

Problem 8 In this case the electric and magnetic forces are in the same direction, so Eq. 6.3 is modified to become

$$F = \left(1 + \frac{v}{c}\right)\frac{e^2 N}{2\pi\varepsilon_0 r} \approx \frac{2e^2 N}{2\pi\varepsilon_0 r}$$

Comparison of this expression with Eq. 6.19 leads to the result of Part (b) after setting $r = d$. The long-range tune shift differs in sign between the two transverse degrees of freedom. In a drift space, the amplitude function is a parabola (Problem 11 of Chapter 3), and since the minimum value is at the crossing point by design, $\beta = \beta^* + s^2/\beta^* \approx s^2/\beta^*$. Setting $d = \alpha s$ and using the expression for the emittance yields the result of Part (d). Part (e) combines results of earlier sections under the assumption that it is meaningful to simply add the several tune shifts. The numerical example gives 2×10^{-4} for the long-range tune shift.

Problem 9 In cylindrical coordinates with the center on the center-line of the cylinder, place one line charge at $r = y,\ \theta = 0$ and its oppositely charged image at $r = R^2/y,\ \theta = 0$. Then the scalar potential is

$$\phi(r,\theta) \propto \ln\left(\frac{r^2 + R^4/y^2 - 2(R^2/y)\cos\theta}{r^2 + y^2 - 2ry\cos\theta}\right)$$

and the boundary condition that the tangential component of the electric field vanish on the surface of the cylinder is verified by calculating $\partial\phi/\partial\theta$ at $r = R$.

Problem 10 Use of Eq. 6.23 in the calculation of F' for insertion into the first row in Eq. 6.18 gives a tune shift of 0.24 for the parameters stated.

Problems 11, 12 13 This material is discussed in Chapter 2 of Alex Chao's book *Physics of Collective Beam Instabilities in High Energy Accelerators*, which can be downloaded from his website at http://www.slac.stanford.edu/~achao/wileybook.html. The treatment of Problem 11 begins halfway down page 47.

Problem 14 For this problem, the elementary definition of a lumped inductive impedance is enough: $Z = i\omega L$, where the inductance, L is the "flux linkage per unit current".

$$Z = i\omega L = i\omega \int_b^d g\frac{\mu_0}{2\pi r}dr = i\frac{v}{R}g\frac{\mu_0}{2\pi}\ln(d/b)$$

and using $c = 1/(\mu_0\varepsilon_0)^{1/2}$ gives the result.

Problem 15 The modes of the head-tail instabilities are discussed in Sections 4.3 and 4.5 of Alex Chao's book, cited above for Problems 11–13.

Problem 16 In the mid-1980s, while converting the Fermilab HEP program toward colliding beams, the microwave instability interfered with the debunching process in the Main Ring. Substantial modification of the interfaces between magnets was necessary to lower the impedance. The parameters in this problem reflect the situation. The square root of Eq. 6.181 gives an estimate of 72 turns for the growth rate.

Problem 17 These parameters would give a threshold of 2×10^7, a few orders of magnitude less than the actual bunch intensity. But in the 12 msec synchrotron oscillation period there are 10^4 betatron oscillations. Even if the momentum spread were as low as 10^{-4}, which it isn't, a small chromaticity will stabilize the beam against the strong head-tail effect. The head-tail instability as described in Section 6.4.3, however does require careful chromaticity adjustment.

Problems 18–21 These problems are based on the article by H. G. Hereward in CERN 77-13, 19 July 1977. This is the CERN "Yellow Book" representing the proceedings of the First Course of the International School of Particle Accelerators. For the Hereward paper and others in this valuable collection, go to http://library.cern.ch/cern _publications/yellow_reports_about.html and follow the suggested links.

Problem 22 Another example from CERN 77-13, this time from the article by Albert Hofmann. The stability diagram for this problem may be found in Figure 8 on page 156, and the process of its generation is described on the preceding pages.

Problem 23 Combination of Eqs. 6.192, 6.50, and the definition of g_0 from Eq. 6.32 gives for the stability criterion

$$\left(\frac{\delta p}{p}\right)^2 = \frac{F_b N f r_0 \gamma_t^2}{c\gamma^3}\left(1 + 2\ln\frac{b}{a}\right)$$

where F_b stands for the bunching factor, f for the orbit frequency, and $\eta \approx 1/\gamma_t^2$. The result is that the fractional momentum spread should be larger than 10^{-4}, and that is easily the case without the necessity of special measures.

7 Chapter 7

Problem 1 In Eq. 7.25, insert the 39% emittance definition from Table 3.1 as modified to normalized form according to Eq. 3.119 to give

$$\Delta\varepsilon_N = \frac{\pi}{2}\gamma\frac{\Delta x^2}{\beta} = \frac{\pi}{2}\gamma \times 10^{-2} \text{ mm mrad}$$

Problem 2 This is just a variation of Eq. 7.30 obtained from an earlier set of notes differing only in the definition of σ_p. Another oversight in update conversion.

Problem 3 Use J as defined in Eq. 3.75 to verify this relation, as commented in the text on page 237.

Problem 4 The gradient error represents the insertion of a quadrupole of strength $(1/f)\Delta B'/B'$. Use Eq. 3.153 to calculate $\Delta\alpha$, and from that evaluate $\det(\Delta J)$. The emittance ratio follows from Eqs. 7.55 and 7.56.

Problem 5 The relation between ΔJ evaluated at positions 1 and 2 is $\Delta J_2 = M(1,2)\Delta J_1$. The product of the determinants is the determinant of the product, and since $\det M(1,2) = 1$, the determinant of ΔJ is an invariant in this circumstance.

Problem 6 This is another variant of Eq. 7.55 after replacement of the γs using $\gamma \equiv (1+\alpha^2)/\beta$.

Problem 7 The solution amounts to showing that $\Delta\beta/\beta$ as written in Problem 27 of Chapter 3 satisfies the differential equation, using $\det(\Delta J) = -\Delta\alpha^2 = -(q\beta_1)^2$.

Problem 8 Keeping only the lowest non-vanishing phase terms in Eq. 2.45 gives the small amplitude invariant

$$\left(\frac{\Delta E}{E_0}\right)^2 + (\Delta\phi)^2 = \text{constant}; \quad E_0^2 \equiv \frac{E_s eV}{\omega_{\text{rf}}\tau(v/c)^2}\left(-\frac{\cos\phi_s}{\eta}\right).$$

In this case, the analysis used in Eqs. 7.19 – 7.24 can be used. Suppose σ_E is the incoming rms energy spread, and σ_ϕ is the rms phase spread within a bunch. Then,

(a) for a ΔE energy error upon injection,

$$(\sigma_E/E_0)^2 = ((\sigma_E)_0/E_0)^2 + \frac{1}{2}(\Delta E/E_0)^2;$$

(b) for a $\Delta\phi$ phase error upon injection,

$$\sigma_\phi^2 = (\sigma_\phi)_0^2 + \frac{1}{2}\Delta\phi^2.$$

One could then multiply each line by appropriate constants to get the answers in terms of eV-sec, etc ...

Problem 9 The phase space area is conserved but is stretched out into thin ribbons occupying a larger effective area.

Problem 10 The expressions of Table 7.1 relate the variances of the particle distribution before and after dilution takes place. With "emittance" first described as a phase space *area*, the point of the problem was to compare the final phase space area which contains 39% of the particles with the initial phase space area that contained 39% of the particles. Unfortunately, perhaps, a numerical integration is involved...

Start with Eq. 7.16, and integrate out to a radius a which is presumed to contain a fraction f (=39%) of the particles:

$$\begin{aligned}
\int\int n_0(x,p_x)dxdp_x &= \frac{1}{2\pi\sigma_0^2}\int\int e^{-[(x-\Delta x)^2+P_x^2]/2\sigma_0^2}dxdp_x \\
&= \frac{e^{-\Delta x^2/2\sigma_0^2}}{2\pi\sigma_0^2}\int_0^{2\pi}\int_0^a e^{-r^2/2\sigma_0^2}e^{(\Delta x\; r/\sigma_0^2)\cos\theta}rdrd\theta \\
&= e^{-\Delta x^2/2\sigma_0^2}\int_0^{a/\sigma_0} e^{-u^2/2}\; I_0\left(\frac{\Delta x}{\sigma_0}u\right)u\; du \\
&= f \;=\; 0.39.
\end{aligned}$$

Here, we use the fact that $I_0(z) = (1/2\pi)\int_0^{2\pi} e^{z\cos\theta}d\theta$. For a given value of $\delta \equiv \Delta x/\sigma_0$, we wish to solve for $d \equiv a/\sigma_0$ which satisfies

$$\int_0^d e^{-u^2/2}\; I_0(\delta \cdot u)u\; du = f \cdot e^{\delta^2/2}.$$

The actual phase space dilution factor, $d^2 = (a/\sigma_0)^2$ for a given δ is then to be compared with the result $1 + \delta^2/2$ for the increase in variance.

Below is a short "code" written in `R`[1] which solves this equation for $0 < \delta < 2$ and generates the plot shown in the accompanying figure. We see that the actual phase space area is underestimated by the ratio of the variances for $\Delta x/\sigma_0$ above unity.

```
# Program to compute phase space area containing a
# fraction f of the beam particles resulting from an
```

[1]The `R` Project for Statistical Computing, http://www.r-project.org/ .

```
# injection offset of amount dx

# dx is offset, in units of sigma0
# df is phase space area, containing the fraction
#  "f" of the particles, in units of sigma0^2

dx <- array(0,100)
df <- array(0,100)

f <- 1-exp(-1/2) # should be ~39%

dd <- 0
# function to integrate
fcn <- function(u){exp(-(u)^2/2)*besselI(u*dd,0)*u }
# function for performing the integral
intgl <- function(d){ integrate(fcn,0,d)$value }
# find root of following, given an offset "dd"...
eqslv <- function(d){ intgl(d)-f*exp(dd^2/2) }

for(i in 1:100){
dd <- dd + 1/50
answr <- uniroot(eqslv, c(0, 10))$root ;
dx[i] <- dd
df[i] <- answr*answr # phase space area
}

plot(dx,df,ylim=c(0,max(df)),typ="l",xlab="dx/sigma",
   ylab="dilution factor")
curve(1+x^2/2,add=TRUE, lty=2)
# last curve is the "variance" of the
#    resulting phase space distribution
#    for comparison
```

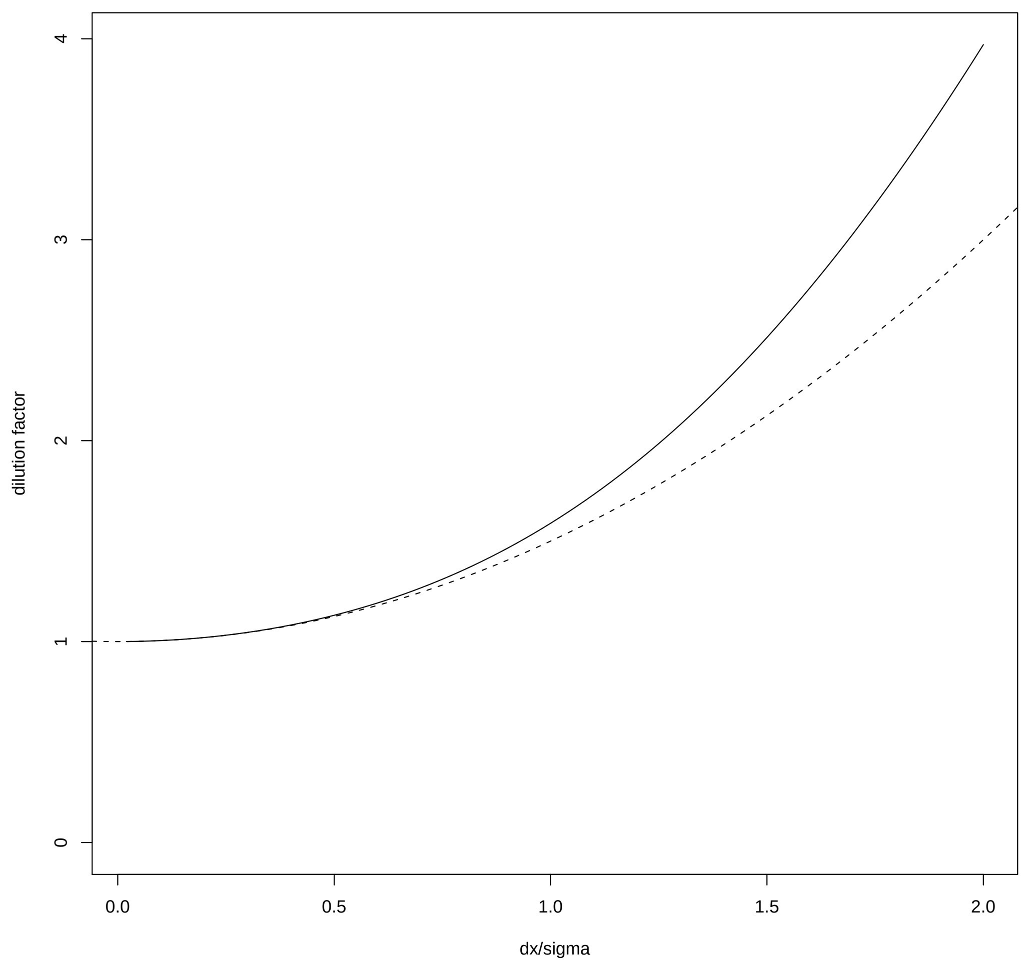

Figure 1: Solid line = phase space area containing 39% of particles after dilution; dashed line = increase in variance of the distribution after dilution.

Problem 11 Here, one needs to make use of relationships among derivatives of Bessel functions, namely

$$J_0'(u) = -J_1(u), \qquad \text{and} \qquad uJ_1'(u) + J_1(u) = uJ_0(u); \qquad ' \equiv d/du.$$

Though not asked to show in the problem, for completeness the expression for the coefficients c_n, Eq. 7.78, comes from the orthogonality relationship:

$$\begin{aligned}\int_0^1 J_0(\lambda_n u)J_0(\lambda_m u)u\,du &= 0, \qquad\qquad m \neq n \\ &= \frac{1}{2}\left[J_1(\lambda_n)\right]^2, m = n,\end{aligned}$$

where $J_0(\lambda_n) = J_0(\lambda_m) = 0$.

Problem 12 We use Eq. 7.131 and the definition of W_a (just below Eq. 7.73) to determine the value of τ at time $t = 4$ sec. Also, for each case, we need to know the initial value of σ/a. The initial value of σ is determined from the emittance value given in the problem. So, we find...

(a),

$$\begin{aligned}\tau &= \left(\frac{R}{W_a}\right)t = \left(\frac{\langle\beta\rangle}{a}\right)^2 (3.3\times 10^{-7}/\text{sec})\frac{P[\mu\text{torr}]}{\gamma} \\ &= \left(\frac{50}{0.01}\right)^2\left(\frac{6.6\times 10^{-7}}{\gamma}\right) = \frac{16.5}{\gamma} = 1.73\end{aligned}$$

for a proton beam with kinetic energy of 8 GeV. The initial beam size is given by $\sigma = \sqrt{\beta\epsilon_N/(6\pi\gamma v/c)}$, which here is about 3.6 mm for our numbers. Thus, $\sigma/a \approx 0.4$ and Figure 7.9(a) roughly applies. We see that at $\tau \approx 1.7$, only about 10% or less of the particles have survived.

(b),
Replace $\gamma = 9.5$ with $\gamma = 22$ above, to find $\tau \approx 0.75$, and $\sigma/a \approx 0.24$. So, Figure 7.8(b) might be more appropriate, indicating that roughly half the particles would be lost in 4 seconds in this case.

Problem 13 If the initial particle distribution is uniform out to a radius a_0 in phase space, where the limiting aperture is defined by a, then the distribution function at time $\tau = 0$ must be

$$f_0(Z,0) = \frac{a^2}{a_0^2}, \qquad Z \leq a_0^2/a^2$$

and zero from $a_0^2/a^2 < Z \leq 1$. To show the two expressions in this problem, one needs only to determine the coefficients c_n:

$$\begin{aligned}c_n &= \frac{1}{J_1(\lambda_n)^2}\int_0^1 f_0(Z)J_0(\lambda_n\sqrt{Z})dZ \\ &= \frac{a^2}{a_0^2}\frac{1}{J_1(\lambda_n)^2}\int_0^{a_0^2/a^2} J_0(\lambda_n\sqrt{Z})dZ\end{aligned}$$

$$
\begin{aligned}
&= 2\frac{a^2}{a_0^2}\frac{1}{J_1(\lambda_n)^2}\int_0^{a_0/a} J_0(\lambda_n u)u\,du \\
&= 2\frac{a^2}{a_0^2}\frac{1}{J_1(\lambda_n)^2}\frac{1}{\lambda_n^2}\left[J_1(\lambda_n u)\cdot\lambda_n u\right]_0^{a_0/a} \\
&= 2\frac{a}{a_0}\frac{J_1(\lambda_n a_0/a)}{\lambda_n J_1(\lambda_n)^2}
\end{aligned}
$$

which, when substituted into Eqs. 7.77 and 7.90, give the desired results.

Problem 14 Use the Booster parameters of the appendix. The first passage through the foil strips the electrons from the negative hydrogen ions to inject the protons. The remaining passages ideally should not take place, but represent the time for the orbit manipulation to take the circulating beam away from the foil. The radiation length in carbon is 18.5 cm. At a kinetic energy of 200 MeV, $pv = 365$ MeV. In order to minimize emittance dilution the foil would be placed at a location of small β. Seeing that the maximum of the amplitude function from the table in the appendix is about 40 m, take 10 m for this estimate. The product $\gamma(v/c)\beta\langle\theta^2\rangle \approx 1/10$ mm-mrad, for a single passage. The emittance dilution discussions in the text suppose that much time elapses following the initial event; such is not the case here. So for the several subsequent passages through the foil, we should allow dilution of about 1 mm-mrad, still acceptable.

Problem 15 Combining Eqs. 1.23 and 7.112 gives

$$\frac{e^2\langle v^2\rangle}{E^2} = \frac{1}{3\pi\gamma f_0\mathcal{H}}\frac{\varepsilon_N}{\tau}$$

where ε_N is the 95% version of the normalized emittance. After calculating $\mathcal{H}$ according to Eq. 7.106 and setting $E = 1$ TeV, the estimate for $ev_{\rm rms}$ is 0.58 MeV. Since the maximum amplitude of the Tevatron RF system is just a few MeV, the emittance growth must be due to another mechanism.

Problem 16 According to Eq. 7.153, optimum cooling will occur for $g = 1/(M+U)$. From Eq. 7.151, $M = 8.6$. The time constant τ is then about 27 msec.

Problem 17 Since the transverse components of momentum are equal in the two frames, the expressions for T_x in part (a) follow from $p_{x0} = p_x = \gamma mv$ and the 39% emittance definition. The temperature of part(b) is about 25 eV. For part (c), $\Delta p_s = p_{s0}/\gamma$ and the temperature is $10^3/3$ eV. In contrast, electrons from a cathode at 1000 K would have a "temperature" of about 0.1 eV.

Problem 18 Adaptation of the process leading to Eq. 7.112 gives the result

$$\frac{d\varepsilon_s}{dt} = \frac{1}{2}\left[\frac{1}{(v/c)eVE\omega_{\rm rf}\tau}\left(-\frac{\eta}{\cos\phi_s}\right)\right]^{1/2} e^2\langle v^2\rangle$$

8 Chapter 8

Problem 1 With use of Eqs. 8.11 and 8.12, the answer for part (a) is 8.85 MeV per turn. Taking 6400 km as the radius of the earth, the result for part (b) is 8.6 TeV per turn.

Problem 2 Scaling down Eq. 8.12 by the 4th power of the proton-electron mass ratio gives 173 eV for the energy radiated per turn. Multiplication by the 70 mA current gives a radiated power of 12 watts. An ideal Carnot engine operating between 4K and 300K would require 75 watts to remove 1 watt, so the 20% efficient device in the problem would need 4.5 kW input power.

Problem 3 The damping times in terms of τ_0 appear in Eqs. 8.21, 8.31, and 8.36. For the Main Ring, $\mathcal{D} \approx 0.03$ and so can be neglected for this estimate. The characteristic time τ_0 is 17 msec.

Problem 4 This problem involves somewhat more algebra than most of the others and actually gets a bit into accelerator design. First relate the parameter K that appears in just about all of the equations of motion in Chapter 3 to something that can be easily visualized. Within a bending magnet that also provides the focusing, the ratio B'/B has the dimensions of an inverse length, and let's call that length x_0. If you imagine that the magnet shown in the figure for Problem 11 of Chapter 1 has its poles inclined so as to produce a field gradient, then x_0 is the distance from the magnet midpoint to the projected intersection of the pole profiles. In these terms, one can write

$$K = \frac{1}{\rho x_0}$$

where ρ is a rather large length appropriate to the scale of the accelerator and, as noted above, x_0 is a smaller quantity related to the magnet size, and therefore the beam aperture. Design desires (at least following the invention of strong focusing) want the beam to be localized within a region of a few centimeters and magnet designs to accede to that wish are quite reasonable with x_0 in the region of one-third to one-half meter. According to Eq. 8.36, horizontal betatron oscillations will exhibit growth with time due to synchrotron radiation provided $\mathcal{D} > 1$.

Since the radius of curvature, ρ is a constant in this case, we have

$$\mathcal{D} = \frac{1}{\rho}\langle D\rangle + \frac{2}{x_0}\left[\langle D\rangle_f - \langle D\rangle_d\right]$$

where the subscripts on the averages over the dispersion refer to the radially focusing and defocusing magnets. Since the dispersion is designed to be much less than the radius of curvature, the first term in the equation above will contribute little, so the interest is on the second term of the equation. Let D_0, D_0' represent the dispersion and its slope at entry to the focusing magnet. The solution for Eq. 3.128 then takes the form

$$\begin{pmatrix} D_0 \\ D_0' \end{pmatrix} = (I - M)^{-1} \begin{pmatrix} e \\ f \end{pmatrix} = \frac{I - M^{-1}}{4\sin^2(\mu/2)} \begin{pmatrix} e \\ f \end{pmatrix}$$

where M is the 2-by-2 matrix for the system, μ is the phase advance through a cell, and e, f have the significance shown in Eq. 3.129.

In order to carry out the averages, one needs the solution for D throughout the magnets. That means solving Eq. 3.127 and don't forget the term on the right hand side. That means, for example, that the solutions through the horizontally focusing and defocusing magnets are, respectively, of the form

$$\begin{aligned} D_f &= x_0 + A_f \cos(\sqrt{K}z) + B_f \sin(\sqrt{K}z) \\ D_d &= -x_0 + A_d \cosh(\sqrt{K}z) + B_d \sinh(\sqrt{K}z) \end{aligned}$$

where the A, B coefficients are to be found in terms of D_0, D_0' as determined above. The remainder of the problem is just a matter of sorting out the trigonometric and hyperbolic functions. The result is

$$\mathcal{D} = 2\frac{\sinh(\sqrt{K}L) + \sin(\sqrt{K}L)}{\sqrt{K}L} \geq 4$$

The construction of the earliest electron synchrotrons was not carried out with these stability considerations in mind. But when beam storage became the priority, the design process changed immediately.

Problem 5 The ratio of Eq. 8.56 to Eq. 8.55 is, according to Eq. 8.54, γ^3. For the un-normalized emittance one factor of γ cancels on both sides of Eq. 8.45. But, after examination of Eq. 8.47, it is the variance of the momentum spread, not the fractional momentum spread, that is proportional to the square of the energy in contrast to the statement of the problem.

Problem 6 The discussion leading to Eq. 8.48 would produce a factor of $(1 - v/c) \approx 1/(2\gamma^2)$ to multiply the undulator period length. An additional term is needed to take into account the wobbly trajectory in arriving at the standard undulator formula.

Problem 7 For particle separation large compared with the wavelength of interest and sufficiently uncorrelated motion, scaling according to n works. Recently, particle densities have been high enough that "coherent synchrotron radiation" (CSR) has become important; this radiation varies as n^2 like the I^2 of antennas.

Problem 8 The fields at the edge of the cylindrical bunch of radius r and length L will produce a transverse force on the oncoming particle of magnitude

$$F = \frac{ne^2}{2\pi\epsilon_0 Lr}(1 + v^2/c^2)$$

which, when integrated over the time of passage, yields a change in transverse momentum

$$\Delta p_x = \frac{2ne^2}{2\pi\epsilon_0 Lr}\left(\frac{1}{2}\frac{L}{c}\right) = \frac{ne^2}{2\pi\epsilon_0 rc} = \frac{2nr_0 mc^2}{rc}.$$

Thus, the deflection angle of the oncoming particle will be

$$\Delta\theta = \frac{\Delta p_x}{p} = \frac{2n}{\gamma} \cdot \left(\frac{r_0}{r}\right).$$

The critical energy, given by Eq. 8.54, when compared to the particle's total energy is then

$$\begin{aligned}
\frac{w_c}{E} &= \frac{3}{2}\gamma^3 \frac{\hbar}{E}\frac{\Delta\theta}{\Delta t} \\
&= \frac{3}{2}\frac{\gamma^3\hbar}{E}\left(\frac{2n}{\gamma}\ \frac{r_0}{r}\right)\left(\frac{2c}{L}\right) \\
&= 6\gamma n\, \frac{\hbar c}{mc^2 L} \cdot \frac{r_0}{r} \\
&= 6(10^7)(10^9)\left(\frac{197 \times 10^{-15}}{(0.511)(10^{-6})}\right)\frac{2.8 \times 10^{-15}}{10^{-6}} \\
&= 6.5 \times 10^4.
\end{aligned}$$